

# Contents

<b>Acknowledgements</b>	<b>i</b>
<b>Abstract (English/Français)</b>	<b>iii</b>
<b>Notation</b>	<b>xi</b>
<b>1 Introduction</b>	<b>1</b>
<b>I Discontinuous Galerkin multiscale methods for advection-diffusion multi-scale PDEs</b>	<b>9</b>
<b>2 Heterogeneous multiscale methods (HMM) for elliptic diffusion problems</b>	<b>13</b>
2.1 Basic results from homogenization theory . . . . .	14
2.2 Singlescale finite element methods . . . . .	17
2.2.1 The classical finite element method (FEM) . . . . .	17
2.2.2 The discontinuous Galerkin finite element method (DG-FEM) . . . . .	19
2.3 Review of numerical homogenization methods . . . . .	21
2.3.1 Solving the finescale problem using modified finite elements . . . . .	21
2.3.2 Solving an effective equation with standard finite elements . . . . .	24
2.4 HMMs based on macro and micro finite element solvers . . . . .	25
2.4.1 The finite element heterogeneous multiscale method (FE-HMM) . . . . .	25
2.4.2 The discontinuous Galerkin finite element heterogeneous multiscale method (DG-HMM) . . . . .	28
2.5 A priori error estimates for FE-HMM and DG-HMM . . . . .	29
2.6 Numerical results for higher order DG-HMM . . . . .	32
<b>3 The DG-HMM for elliptic advection-diffusion problems</b>	<b>35</b>
3.1 Model problem and homogenization results . . . . .	38
3.2 The DG-HMM . . . . .	40
3.2.1 Preliminaries . . . . .	40
3.2.2 DG-HMM for advection-diffusion problems . . . . .	42
3.2.3 A useful reformulation of the DG-HMM . . . . .	45
3.3 Main results . . . . .	47
3.3.1 Stability results . . . . .	48

3.3.2	A priori error estimates . . . . .	49
3.4	Proof of the stability results . . . . .	52
3.4.1	Preliminaries . . . . .	52
3.4.2	Inf-sup condition and stability result . . . . .	56
3.5	Proof of the a priori error estimates . . . . .	61
3.5.1	Semi-discrete error . . . . .	61
3.5.2	Fully discrete error . . . . .	63
3.5.3	Proofs of the modeling error estimates . . . . .	65
3.6	Numerical results . . . . .	70
3.6.1	Convergence rates . . . . .	70
3.6.2	Advection dominated multiscale problem with a boundary layer . . . . .	73
3.6.3	Example with non-periodic, random data . . . . .	75
3.7	The effect of numerical integration for singlescale DG-FEM . . . . .	76
3.7.1	DG-FEM without numerical quadrature . . . . .	77
3.7.2	DG-FEM with numerical quadrature . . . . .	77
3.7.3	Numerical results for singlescale DG-FEM . . . . .	79
3.8	Summary . . . . .	83
<b>4</b>	<b>The DG-HMM for parabolic advection-diffusion problems with large drift</b>	<b>85</b>
4.1	Model problem and its homogenization . . . . .	88
4.2	Multiscale method . . . . .	92
4.2.1	Spatial macro and micro discretizations . . . . .	92
4.2.2	Space-discrete DG-HMM . . . . .	94
4.2.3	Existence and uniqueness of the numerical solution . . . . .	96
4.3	Main results . . . . .	99
4.3.1	A priori error estimates for macro spatial error . . . . .	101
4.3.2	Fully discrete analysis of spatial macro and micro errors . . . . .	102
4.4	Proof of the main results . . . . .	104
4.4.1	Preliminaries . . . . .	104
4.4.2	Error propagation formula . . . . .	107
4.4.3	Spatial macro error . . . . .	107
4.4.4	Spatial micro error . . . . .	115
4.5	Numerical results . . . . .	123
4.5.1	Convergence rates of spatial errors . . . . .	123
4.5.2	Comparison between HMM solution and finescale solution . . . . .	125
4.6	Discussion of alternative upscaling strategies . . . . .	130
4.6.1	A seamless upscaling procedure . . . . .	130
4.6.2	Non-periodic micro-macro coupling conditions . . . . .	132
4.7	Summary . . . . .	135
<b>5</b>	<b>Conclusion and Outlook of Part I</b>	<b>137</b>
5.1	Outlook . . . . .	137

<b>II Finite element heterogeneous multiscale methods for nonlinear monotone multiscale PDEs</b>	<b>141</b>
<b>6 The FE-HMM for nonlinear monotone problems</b>	<b>145</b>
6.1 Literature overview . . . . .	145
<b>Parabolic problems</b>	<b>147</b>
6.2 Model problem and homogenization . . . . .	148
6.3 Multiscale method . . . . .	150
6.3.1 FE-HMM for nonlinear monotone parabolic problems . . . . .	150
6.3.2 A useful reformulation of the FE-HMM . . . . .	152
6.3.3 Existence and uniqueness of the numerical solution . . . . .	153
6.4 Main results . . . . .	156
6.4.1 Estimates for temporal and spatial macro errors . . . . .	156
6.4.2 Fully discrete space-time a priori error estimates . . . . .	157
6.5 Proof of the main results . . . . .	160
6.5.1 Preliminaries . . . . .	160
6.5.2 Error propagation formula . . . . .	165
6.5.3 Temporal and macro spatial errors . . . . .	165
6.5.4 Explicit estimates for the HMM upscaling error $r_{HMM}$ . . . . .	169
6.6 Implementation and numerical results . . . . .	177
6.6.1 Implementation . . . . .	177
6.6.2 Convergence rates . . . . .	178
6.6.3 Influence of the sampling domain size $\delta$ . . . . .	179
6.7 Summary . . . . .	181
<b>Elliptic problems</b>	<b>183</b>
6.8 FEM for nonlinear monotone elliptic PDEs . . . . .	185
6.8.1 FEM with numerical integration . . . . .	185
6.8.2 A linear elliptic projection . . . . .	186
6.8.3 Optimal $L^2$ error estimate for FEM without quadrature formula . . . . .	188
6.8.4 A priori error estimate for FEM with quadrature formula . . . . .	189
6.9 FE-HMM for nonlinear monotone elliptic multiscale problems . . . . .	193
6.9.1 Multiscale method . . . . .	194
6.9.2 Fully discrete a priori error estimates. . . . .	195
6.9.3 Proof of the error estimates . . . . .	197
6.10 Summary . . . . .	199
<b>7 A linearized FE-HMM for nonlinear monotone parabolic problems</b>	<b>201</b>
7.1 Model problem and homogenization . . . . .	203
7.2 Nonlinear and linearized multiscale methods . . . . .	206
7.2.1 Micro and macro finite element spaces . . . . .	206
7.2.2 Nonlinear FE-HMM . . . . .	207

7.2.3	Linearized FE-HMM . . . . .	207
7.3	Main results . . . . .	209
7.3.1	Well-posedness of the numerical method . . . . .	209
7.3.2	A priori error estimates . . . . .	211
7.4	Analysis . . . . .	216
7.4.1	Preliminaries . . . . .	216
7.4.2	Proof of the a priori error estimates . . . . .	220
7.5	Numerical results . . . . .	228
7.5.1	Convergence rates and performance comparisons . . . . .	228
7.5.2	Case of a degenerated problem . . . . .	236
7.6	Summary . . . . .	240
<b>8</b>	<b>The FE-HMM for nonlinear monotone parabolic problems in the <math>W^{1,p}</math> setting</b>	<b>241</b>
8.1	Homogenization of the model problem . . . . .	242
8.2	Multiscale method . . . . .	244
8.2.1	FE-HMM for nonlinear monotone parabolic problems . . . . .	245
8.2.2	A useful reformulation of the FE-HMM . . . . .	246
8.2.3	Existence and uniqueness of the numerical solution . . . . .	247
8.3	Main results . . . . .	252
8.4	Proof of the main results . . . . .	255
8.4.1	Error propagation formula . . . . .	255
8.4.2	Difference between weak and strong formulation in time . . . . .	256
8.4.3	Temporal and macro spatial errors . . . . .	257
8.4.4	Abstract estimates for the HMM upscaling error . . . . .	260
8.4.5	Proof of the general convergence result . . . . .	261
8.5	Numerical results: Simulation of a laminated iron core . . . . .	264
8.6	Summary . . . . .	267
<b>9</b>	<b>Conclusion and Outlook of Part II</b>	<b>269</b>
9.1	Outlook . . . . .	269
	<b>Bibliography</b>	<b>283</b>
	<b>Curriculum Vitae</b>	<b>285</b>